

# Square roots of perturbed subelliptic operators on Lie groups

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**ABSTRACT.** We solve the Kato square root problem for bounded measurable perturbations of subelliptic operators on connected Lie groups. The subelliptic operators are divergence form operators with complex bounded coefficients, which may have lower order terms. In this general setting we deduce inhomogeneous estimates. In case the group is nilpotent and the subelliptic operator is pure second order, then we prove stronger homogeneous estimates. Furthermore, we prove Lipschitz stability of the estimates under small perturbations of the coefficients.

## 1 Introduction

The Kato problem in  $\mathbb{R}^d$  was a long standing problem which was solved by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [AHLMT] in 2002. The papers of Hofmann [Hof] and McIntosh [McI2] and the book by Auscher and Tchamitchian [AT] provide a narrative of the resolution of Kato's conjecture. This problem was recast in terms of the functional calculus of a first-order system by Axelsson, Keith and McIntosh in [AKM1] and [AKM2], which together provide a unified first-order framework for recovering and extending some results concerning the harmonic analysis of strongly elliptic operators. A version of the Kato problem was presented by Morris for manifolds with exponential growth in [Mor], and another version on metric measure spaces with the doubling property was presented by Bandara in [Ban]. The main aim of this paper is to present a solution to the Kato problem for subelliptic operators on Lie groups.

Let  $G$  be a connected Lie group with Lie algebra  $\mathfrak{g}$  and (left) Haar measure  $\mu$ . All integration on  $G$  is with respect to the (left) Haar measure and the norm is the  $L_2$ -norm, unless stated otherwise. Let  $a_1, \dots, a_m$  be an algebraic basis for  $\mathfrak{g}$ , that is, an independent set which generates  $\mathfrak{g}$ . Let  $L$  be the left regular representation in  $L_2(G)$ . So  $(L(x)f)(y) = f(x^{-1}y)$  for all  $x \in G$ ,  $f \in L_2(G)$  and a.e.  $y \in G$ . For all  $k \in \{1, \dots, m\}$  let  $A_k$  be the infinitesimal generator of the one-parameter unitary group  $t \mapsto L(\exp ta_k)$ . Then  $A_k$  is skew-adjoint. Define the Sobolev space  $W'_{1,2}(G) = \bigcap_{k=1}^m D(A_k)$  with norm such that

$$\|f\|_{W'_{1,2}(G)}^2 = \|f\|^2 + \sum_{k=1}^m \|A_k f\|^2.$$

Then  $W'_{1,2}(G)$  is a Hilbert space since  $A_k$  is closed for all  $k$ . Note that  $W'_{1,2}(G)$  depends on the choice of the algebraic basis and if confusion is possible, then we write  $W'_{1,2}(G, a)$ . Clearly  $C_c^\infty(G) \subset W'_{1,2}(G)$ , so  $W'_{1,2}(G)$  is dense in  $L_2(G)$ .

Next, for all  $k, l \in \{1, \dots, m\}$  let  $b_{kl}, b_k, b'_k, b_0 \in L_\infty(G)$ . Assume there exists a constant  $\kappa > 0$  such that the following Gårding inequality holds:

$$\begin{aligned} \operatorname{Re} \left( \sum_{k,l=1}^m (b_{kl} A_l u, A_k u) + \sum_{l=1}^m (b_l A_l u, u) + \sum_{k=1}^m (b'_k u, A_k u) + (b_0 u, u) \right) \\ \geq \kappa \left( \sum_{k=1}^m \|A_k u\|^2 + \|u\|^2 \right) \end{aligned} \quad (1)$$

for all  $u \in W'_{1,2}(G)$ . Then the inhomogeneous divergence form operator

$$H_I = - \sum_{k,l=1}^m A_k b_{kl} A_l + \sum_{l=1}^m b_l A_l - \sum_{k=1}^m A_k b'_k + b_0 I$$

is a maximal accretive operator on  $L_2(G)$  of type  $\omega$  for some  $\omega \in [0, \frac{\pi}{2})$ . So  $-H_I$  generates a bounded semigroup on  $L_2(G)$ .

It is easy to see that (1) holds under the homogeneous Gårding inequality

$$\operatorname{Re} \sum_{k,l=1}^m (b_{kl} A_l u, A_k u) \geq \kappa \sum_{k=1}^m \|A_k u\|^2 \quad (2)$$

with a possible change in  $\kappa$ , provided a large enough positive constant is added to  $b_0$ . For example, if

$$\operatorname{Re} \sum_{k,l=1}^m b_{kl} \xi_k \bar{\xi}_l \geq \kappa |\xi|^2$$

a.e. for all  $\xi \in \mathbb{C}^m$ , then (2) is valid.

Furthermore, let  $b \in L_\infty(G)$  and suppose there exists a constant  $\kappa_1 > 0$  such that  $\operatorname{Re} b \geq \kappa_1$  a.e. Then  $bH_I$  is an  $\tilde{\omega}$ -sectorial operator on  $L_2(G)$  for some  $\tilde{\omega} < \pi$ , so it has a unique square root  $\sqrt{bH_I}$  which is  $\frac{1}{2}\tilde{\omega}$ -sectorial and satisfies  $(\sqrt{bH_I})^2 = bH_I$ .

The main theorems of this paper are as follows. The first one is the solution of the inhomogeneous Kato problem for subelliptic operators.

**Theorem 1.1** *Let  $G$  be a connected Lie group and suppose that  $a_1, \dots, a_m$  is an algebraic basis for the Lie algebra  $\mathfrak{g}$  of  $G$ . Let*

$$H_I = - \sum_{k,l=1}^m A_k b_{kl} A_l + \sum_{l=1}^m b_l A_l - \sum_{k=1}^m A_k b'_k + b_0 I$$

*be a divergence form operator with bounded measurable coefficients satisfying the ellipticity condition (1). Let  $b \in L_\infty(G)$  and suppose there exists a constant  $\kappa_1 > 0$  such that  $\operatorname{Re} b \geq \kappa_1$  a.e. Then  $D(\sqrt{bH_I}) = W'_{1,2}(G)$  and there exist  $c, C > 0$  such that*

$$c(\|\sqrt{bH_I}u\| + \|u\|) \leq \|u\| + \sum_{k=1}^m \|A_k u\| \leq C(\|\sqrt{bH_I}u\| + \|u\|)$$

*for all  $u \in W'_{1,2}(G)$ .*

For connected nilpotent Lie groups, or more generally, for Lie groups  $G$  which are the local direct product of a connected compact Lie group  $K$  and a connected nilpotent Lie group  $N$ , a homogeneous result is also valid. (To say that  $G$  is the local direct product of  $K$  and  $N$  means that  $G = K \cdot N$  and  $K \cap N$  is discrete. Equivalently, the Lie algebra of  $G$  is the direct product of the Lie algebra of  $K$  and the Lie algebra of  $N$ .)

**Theorem 1.2** *Let  $G$  be the local direct product of a connected compact Lie group and a connected nilpotent Lie group. Let*

$$H = - \sum_{k,l=1}^m A_k b_{kl} A_l$$

*be a homogeneous divergence form operator with bounded measurable coefficients satisfying the subellipticity condition (2). Let  $b \in L_\infty(G)$  and suppose there exists a constant  $\kappa_1 > 0$  such that  $\operatorname{Re} b \geq \kappa_1$  a.e. Then  $D(\sqrt{b}H) = W'_{1,2}(G)$  and there exist  $c, C > 0$  such that*

$$c \|\sqrt{b}Hu\| \leq \sum_{k=1}^m \|A_k u\| \leq C \|\sqrt{b}Hu\|$$

*for all  $u \in W'_{1,2}(G)$ .*

Lie groups of the above mentioned type necessarily satisfy the doubling property. We do not know whether the doubling property itself implies homogeneous bounds, except in the special case when the algebraic basis  $a_1, \dots, a_m$  is actually a vector space basis. In other words, when  $H$  is strongly elliptic rather than subelliptic, then the conclusion of Theorem 1.2 does hold on all Lie groups with polynomial growth. For vector space bases we drop the prime and write  $W_{1,2}(G) = W'_{1,2}(G)$  and  $W_{1,2}(G, a) = W'_{1,2}(G, a)$ .

**Theorem 1.3** *Let  $G$  be a connected Lie group with polynomial growth and suppose that  $a_1, \dots, a_m$  is a vector space basis for the Lie algebra  $\mathfrak{g}$  of  $G$ . Let*

$$H = - \sum_{k,l=1}^m A_k b_{kl} A_l$$

*be a strongly elliptic homogeneous divergence form operator with bounded measurable coefficients satisfying the ellipticity condition (2). Let  $b \in L_\infty(G)$  and suppose there exists a constant  $\kappa_1 > 0$  such that  $\operatorname{Re} b \geq \kappa_1$  a.e. Then  $D(\sqrt{b}H) = W_{1,2}(G)$  and there exist  $c, C > 0$  such that*

$$c \|\sqrt{b}Hu\| \leq \sum_{k=1}^m \|A_k u\| \leq C \|\sqrt{b}Hu\|$$

*for all  $u \in W_{1,2}(G)$ .*

In Section 3 we prove the homogeneous bounds, first those in Theorem 1.2. The algebraic basis provides a canonical distance  $d$  that is well suited for the study of subelliptic operators. Then  $(G, d, \mu)$  is a metric measure space. The proof is achieved by building upon the results of Bandara [Ban] who adapted the earlier framework of [AKM1] to the situation of metric measure spaces.

The situation in Theorem 1.3 requires more substantial innovations in the proof, involving the structure theory of Lie groups as developed by Dungey–ter Elst–Robinson [DER]. Note that we cannot directly apply homogeneous estimates for second-order derivatives of the form  $\|A_k A_l u\| \leq C\|\Delta u\|$  where  $\Delta$  is the (sub-)Laplacian, as typically used in proofs of the Kato estimates. Indeed these homogeneous estimates  $\|A_k A_l u\| \leq C\|\Delta u\|$  for second-order derivatives are *false* for  $G$ , if  $G$  is a connected Lie group with polynomial growth which is not a local direct product of a connected compact Lie group and a connected nilpotent Lie group. (See [ERS] Theorem 1.1.)

In Section 4 we turn to the proof of the inhomogeneous estimates as stated in Theorem 1.1. The Haar measure is at most exponential in volume growth of balls, so we are in a position where we can adapt the results of Morris [Mor] who obtained Kato estimates on complete Riemannian manifolds which satisfy a similar exponential growth condition on the volume of balls. Indeed his methods work for an arbitrary Borel-regular measure.

In Section 5 we consider some variants of the inhomogeneous results, while in Section 6 we state and prove a Lipschitz estimate of the form (in the homogeneous case)

$$\|\sqrt{(b + \tilde{b})\tilde{H}u} - \sqrt{bHu}\| \leq C(\|\tilde{b}\|_\infty + \sum_{j,k} \|\tilde{b}_{jk}\|_\infty) \|\nabla u\| ,$$

where

$$H = - \sum_{k,l=1}^m A_k b_{kl} A_l \quad \text{and} \quad \tilde{H} = - \sum_{k,l=1}^m A_k (b_{kl} + \tilde{b}_{kl}) A_l ,$$

under small bounded perturbations  $\tilde{b}, \tilde{b}_{kl}$  of the coefficients.

## 2 Preliminaries

In this section we gather some background material on Lie groups and operator theory that will be used throughout the paper.

### 2.1 Lie groups

We use the notation as in the introduction. In particular,  $a_1, \dots, a_m$  is an algebraic basis for the Lie algebra  $\mathfrak{g}$  of a connected Lie group  $G$ .

For all  $k \in \{1, \dots, m\}$  and  $x \in G$  define  $X_k|_x \in T_x G$  by

$$X_k|_x f = \frac{d}{dt} f((\exp ta_k)x) \Big|_{t=0} .$$

Then  $X_k$  is a smooth right invariant vector field on  $G$ . Note that  $A_k f = -X_k f$  for all  $f \in C_c^\infty(G)$ .

The space  $L_2(G, \mathbb{C}^m)$  has a natural inner product. Define the unbounded operator  $\nabla$  from  $L_2(G)$  into  $L_2(G, \mathbb{C}^m)$  by  $D(\nabla) = W'_{1,2}(G)$  and

$$(\nabla u)(x) = ((A_1 u)(x), \dots, (A_m u)(x))$$

for a.e.  $x \in G$ . Then  $\nabla$  is densely defined and closed, since  $A_k$  is closed for all  $k \in \{1, \dots, m\}$ . We denote its adjoint by  $-\text{div}$ . Thus  $\text{div} = -\nabla^*$ .

Define the sesquilinear form  $J: W'_{1,2}(G) \times W'_{1,2}(G) \rightarrow \mathbb{C}$  by

$$J[f, g] = (\nabla f, \nabla g).$$

Then  $J$  is a closed positive symmetric form. Let  $\Delta$  be the self-adjoint operator associated with  $J$ . We call  $\Delta$  the **sub-Laplacian** (associated with the algebraic basis  $a_1, \dots, a_m$ ). Clearly  $\Delta = -\operatorname{div} \nabla$ . There is another simple identity for  $\Delta$ .

**Proposition 2.1** *One has  $\Delta = -\sum_{k=1}^m A_k^2$  with  $D(\Delta) = \bigcap_{k=1}^m D(A_k^2)$ . Moreover,  $D(\Delta) = \bigcap_{k,l=1}^m D(A_k A_l)$  and there exists a constant  $C_1 > 0$  such that*

$$\sum_{k,l=1}^m \|A_k A_l u\|^2 \leq C_1 (\|\Delta u\|^2 + \|u\|^2) \quad (3)$$

for all  $u \in D(\Delta)$ .

Finally, if  $G$  is the local direct product of a connected compact Lie group and a connected nilpotent Lie group, then there exists a constant  $C_2 > 0$  such that

$$\frac{1}{m} \|\Delta u\|^2 \leq \sum_{k,l=1}^m \|A_k A_l u\|^2 \leq C_2 \|\Delta u\|^2$$

for all  $u \in D(\Delta)$ .

**Proof** It is proved in ter Elst–Robinson [ER2] Theorem 7.2.I that  $\Delta_0 := \Delta|_{\bigcap_{k,l=1}^m D(A_k A_l)}$  is a closed operator and in the proof it is shown that  $\Delta_0 = \Delta$  (with same domains). Moreover, ter Elst–Robinson [ER3] Theorem 3.3.III gives  $\bigcap_{k,l=1}^m D(A_k A_l) = \bigcap_{k=1}^m D(A_k^2)$ . Then estimate (3) follows from the closed graph theorem. The final equivalence of seminorms is proved in ter Elst–Robinson–Sikora [ERS] Proposition 4.1.  $\square$

Define

$$W'_{1,2}(G, \mathbb{C}^m) = \{x \mapsto (u_1(x), \dots, u_m(x)) : u_1, \dots, u_m \in W'_{1,2}(G)\} \subset L_2(G, \mathbb{C}^m).$$

Next, the space  $L_2(G, \mathbb{C}^{m^2})$  has a natural inner product. Define the unbounded operator  $\tilde{\nabla}$  from  $L_2(G, \mathbb{C}^m)$  into  $L_2(G, \mathbb{C}^{m^2})$  by  $D(\tilde{\nabla}) = W'_{1,2}(G, \mathbb{C}^m)$  and

$$(\tilde{\nabla} f)(x) = \left( (A_k u_l)(x) \right)_{kl} \quad (4)$$

for a.e.  $x \in G$ , if  $f(x) = (u_1(x), \dots, u_m(x))$  for a.e.  $x \in G$ .

**Remark 2.2** To satisfy our readers with a geometric appetite, we make the following geometric remark. Recall the notion of a connection  $\nabla$  over a vector bundle  $V$ . This is an operator  $\nabla: C_\infty(V) \rightarrow C_\infty(T^*G \otimes V)$  satisfying  $\nabla_{fX}(Y) = f\nabla_X Y$  and  $\nabla_X(fY) = X(f)Y + f\nabla_X Y$  for all  $f \in C_\infty(G)$ ,  $X \in C_\infty(T^*G)$  and  $Y \in C_\infty(V)$ . Now, given a sub-bundle  $E \subset T^*G$ , we can define a **sub-connection**  $\nabla: C_\infty(V) \rightarrow C_\infty(E \otimes V)$  to mean that it satisfies the above properties but with the condition on  $X$  being that  $X \in C_\infty(E)$ . Our philosophy in this paper stems from an observation that we can construct a sub-bundle  $E$  on which our subelliptic operators are strongly elliptic. Since Lie groups are parallelisable, and for the benefit of a wider audience, we refrain from using the language of vector bundles in this paper. However, we refer the reader to [BMc], where the first and third authors provide a solution to a version of the Kato square root problem for certain uniformly elliptic second order operators over a class of vector bundles.

We also need a subelliptic distance on  $G$ . Let  $\gamma: [0, 1] \rightarrow G$  be an absolutely continuous path such that  $\dot{\gamma}(t) \in \text{span}\{X_1|_{\gamma(t)}, \dots, X_m|_{\gamma(t)}\}$  for a.e.  $t \in [0, 1]$ . Define the **length** of  $\gamma$  by

$$\ell(\gamma) = \int_0^1 \left( \sum_{k=1}^m |\gamma^k(t)|^2 \right)^{1/2} dt \in [0, \infty]$$

if  $\dot{\gamma}(t) = \sum_{k=1}^m \gamma^k(t) X_k|_{\gamma(t)}$  for a.e.  $t \in [0, 1]$ . Since  $G$  is connected it follows from a theorem of Carathéodory [Car] that for all  $x, y \in G$  there exists such a path  $\gamma$  with finite length and  $\gamma(0) = x$  and  $\gamma(1) = y$ . If  $x, y \in G$  then we define the **distance**  $d(x, y)$  between  $x$  and  $y$  to be the infimum of the length of all such paths with  $\gamma(0) = x$  and  $\gamma(1) = y$ . Then  $d$  is a metric on  $G$ . For all  $x \in G$  and  $r > 0$ , let  $B(x, r) = \{y \in G : d(x, y) < r\}$ . If  $x = e$ , the identity element of  $G$ , then we write  $B(r) = B(e, r)$ . Then  $B(x, r) = B(r)x$  for all  $x \in G$ .

### Proposition 2.3

- I. *The topology on  $G$  is the same as the topology associated with  $d$ . In particular, the open balls are measurable.*
- II. *The metric space  $(G, d)$  is complete and the closed balls  $\overline{B(x, r)}$  are compact.*
- III. *There exist  $c, C > 0$  and  $D' \in \mathbb{N}$  such that  $cr^{D'} \leq \mu(B(r)) \leq Cr^{D'}$  for all  $r \in (0, 1]$ .*
- IV. *There exist  $C > 0$  and  $\lambda \geq 0$  such that  $\mu(B(r)) \leq Ce^{\lambda r}$  for all  $r \geq 1$ .*

**Proof** For a proof of Statement I, see [VSC] Proposition III.4.1.

Statement II follows from the discussion in Section III.4 in [VSC] and the fact that every locally compact metric space is complete.

Statement III is a consequence of Nagel–Stein–Wainger [NSW] Theorems 1 and 4.

The last statement is proved in Guivarc’h [Gui] Théorème II.3.  $\square$

A **metric measure space**  $(\mathcal{X}, \rho, \nu)$  is a set  $\mathcal{X}$  with a metric  $\rho$  and a measure  $\nu$  on the Borel  $\sigma$ -algebra of  $\mathcal{X}$  induced by the metric on  $\mathcal{X}$ . The metric measure space is said to have the **doubling property** if there exists a constant  $c > 0$  such that  $0 < \nu(B(x, 2r)) \leq c\nu(B(x, r)) < \infty$  for all  $x \in \mathcal{X}$  and  $r > 0$ . It is called **locally exponentially doubling** if there exist  $\kappa, \lambda \geq 0$  and  $C \geq 1$  such that

$$0 < \nu(B(x, tr)) \leq Ct^\kappa e^{\lambda tr} \nu(B(x, r)) < \infty$$

for all  $x \in \mathcal{X}$ ,  $r > 0$  and  $t \geq 1$ . It follows from Proposition 2.3.I that the triple  $(G, d, \mu)$  is a metric measure space. We say the Lie group  $G$  has **polynomial growth** if there exist  $D \in \mathbb{N}$  and  $c > 0$  such that  $\mu(B(r)) \leq cr^D$  for all  $r \geq 1$ . Recall that the **modular function** on  $G$ , which we denote by  $\delta$  throughout this paper, is the function  $\delta: G \rightarrow (0, \infty)$  such that  $\mu(Ux) = \delta(x)\mu(U)$  for every Borel measurable  $U \subset G$  and  $x \in G$ . It is a continuous homomorphism. The group  $G$  is called **unimodular** if  $\delta(x) = 1$  for all  $x \in G$ .

### Proposition 2.4

- I. *The metric measure space  $(G, d, \mu)$  is locally exponentially doubling.*
- II. *The Lie group  $G$  has polynomial growth if and only if the metric measure space  $(G, d, \mu)$  has the doubling property. If  $G$  has polynomial growth then there exist  $c, C > 0$  and  $D \in \mathbb{N}_0$  such that  $cr^D \leq \mu(B(x, r)) \leq Cr^D$  for all  $x \in G$  and  $r \geq 1$ .*

**Proof** ‘I’. It follows from Statements III and IV of Proposition 2.3 that there exist  $C \geq 1$ ,  $\lambda \geq 0$  and  $D' \in \mathbb{N}$  such that  $0 < \mu(B(tr)) \leq C t^{D'} e^{\lambda tr} \mu(B(r))$  for all  $r > 0$  and  $t \geq 1$ . Then for all  $x \in G$ ,  $r > 0$  and  $t \geq 1$  one has

$$\begin{aligned} 0 < \mu(B(x, tr)) &= \mu(B(tr)x) = \delta(x) \mu(B(tr)) \\ &\leq C \delta(x) t^{D'} e^{\lambda tr} \mu(B(r)) = C t^{D'} e^{\lambda tr} \mu(B(x, r)). \end{aligned}$$

So the metric measure space  $(G, d, \mu)$  is locally exponentially doubling.

‘II’. Clearly volume doubling implies polynomial growth. Conversely, suppose that  $G$  has polynomial growth. Then it follows from Guivarc’h [Gui] Théorème II.3 that there exist  $c, C > 0$  and  $D \in \mathbb{N}_0$  such that  $c r^D \leq \mu(B(r)) \leq C r^D$  for all  $r \geq 1$ . Moreover,  $G$  is unimodular by [Gui] Lemma 1.3. Therefore  $\mu(B(x, r)) = \mu(B(r)x) = \mu(B(r))$  for all  $x \in G$  and  $r > 0$  and the volume estimate follows. Together with Proposition 2.3.III it follows that there exists a constant  $c' > 0$  such that  $\mu(B(2r)) \leq c' \mu(B(r))$  for all  $r > 0$ . Therefore the metric measure space  $(G, d, \mu)$  has the doubling property.  $\square$

If  $\eta$  is a scalar valued Lipschitz function  $\eta$  on a metric space  $(\mathcal{X}, \rho)$  without isolated points, then we denote by  $\text{Lip } \eta$  the pointwise Lipschitz constant

$$(\text{Lip } \eta)(x) = \limsup_{y \rightarrow x} \frac{|\eta(x) - \eta(y)|}{\rho(x, y)}$$

whenever  $x \in \mathcal{X}$ .

Let  $L^{(1)}$  be the left regular representation in  $L_1(G)$  and for all  $k \in \{1, \dots, m\}$  let  $A_k^{(1)}$  be the infinitesimal generator of the one-parameter group  $t \mapsto L^{(1)}(\exp t a_k)$ . Let  $A_k^{(\infty)} = -(A_k^{(1)})^*$  be the dual operator on  $L_\infty(G)$ . One has the following relation with Lipschitz functions.

### Proposition 2.5

- I. The space  $\bigcap_{k=1}^m D(A_k^{(\infty)})$  is the space of all bounded Lipschitz functions on  $G$ .
- II. If  $\eta \in \bigcap_{k=1}^m D(A_k^{(\infty)})$ , then  $\sum_{k=1}^m (A_k^{(\infty)} \eta)^2 \leq (\text{Lip } \eta)^2$  a.e.

**Proof** Statement I follows as in Theorem 6.12 in [Hei].

‘II’. Let  $\xi \in \mathbb{R}^m$  with  $|\xi| = 1$ . For all  $t > 0$  define  $y_t = \exp(t \sum_{k=1}^m \xi_k a_k)$ . Then  $d(y_t, e) \leq t$  and if  $t$  is small enough then  $y_t \neq e$ . Let  $x \in G$ . Then for all  $t > 0$  with  $y_t \neq e$  one has

$$\sup_{y \in B(x, t) \setminus \{x\}} \frac{|\eta(y) - \eta(x)|}{d(y, x)} \geq \frac{|\eta(y_t x) - \eta(x)|}{d(y_t x, x)} \geq \frac{|\eta(y_t x) - \eta(x)|}{t} = \left| \frac{1}{t} \left( (L^{(\infty)}(y_t^{-1}) - I) \eta \right)(x) \right|,$$

where  $L^{(\infty)}$  is the left regular representation in  $L_\infty(G)$ . Now choose  $t = \frac{1}{n}$  with  $n \in \mathbb{N}$  and note that

$$\lim_{n \rightarrow \infty} n(L^{(\infty)}(y_{1/n}^{-1}) - I) \eta = - \sum_{k=1}^m \xi_k A_k^{(\infty)} \eta$$

weakly\* in  $L_\infty(G)$ . Therefore

$$|\text{Lip } \eta| \geq \left| \sum_{k=1}^m \xi_k A_k^{(\infty)} \eta \right|$$

a.e. This is for all  $\xi \in \mathbb{R}^m$  with  $|\xi| = 1$ .

Let  $D$  be a countable dense subset of  $\{\xi \in \mathbb{R}^m : |\xi| = 1\}$ . Then there exists a nulset  $N \subset G$  such that

$$|(\text{Lip } \eta)(x)| \geq \left| \sum_{k=1}^m \xi_k (A_k^{(\infty)} \eta)(x) \right| \quad (5)$$

for all  $x \in G \setminus N$  and  $\xi \in D$ . Hence by continuity (5) is valid for all  $x \in G \setminus N$  and  $\xi \in \mathbb{R}^m$  with  $|\xi| = 1$ . Therefore

$$|(\text{Lip } \eta)(x)|^2 \geq \sum_{k=1}^m |(A_k^{(\infty)} \eta)(x)|^2$$

for all  $x \in G \setminus N$ . □

## 2.2 Operator theory

For the convenience of the reader, we shall present some operator theoretic material which sits at the heart of both the homogeneous and inhomogeneous problems.

First, we recall the theory of **bisectorial operators**. For all  $\omega \in [0, \frac{\pi}{2})$  define the **bisector** by

$$S_\omega = \{\zeta \in \mathbb{C} : |\arg \zeta| \leq \omega \text{ or } |\pi - \arg \zeta| \leq \omega \text{ or } \zeta = 0\}$$

and the **open bisector** by

$$S_\omega^o = \{\zeta \in \mathbb{C} \setminus \{0\} : |\arg \zeta| < \omega \text{ or } |\pi - \arg \zeta| < \omega\}.$$

Let  $\mathcal{H}$  be a Hilbert space. An operator  $T: D(T) \rightarrow \mathcal{H}$  with  $D(T) \subset \mathcal{H}$  is then called  **$\omega$ -bisectorial** (or bisectorial with **angle of sectoriality**  $\omega$ ) if it is closed,  $\sigma(T) \subset S_\omega$ , and for each  $\omega < \mu < \frac{\pi}{2}$ , there is a constant  $C_\mu > 0$  such that  $|\zeta| \|(\zeta I - T)^{-1}\| \leq C_\mu$  for all  $\zeta \in \mathbb{C} \setminus \{0\}$  satisfying  $|\arg \zeta| \geq \mu$  and  $|\pi - \arg \zeta| \geq \mu$ .

**Remark 2.6** When  $T$  is  $\omega$ -bisectorial, then  $T^2$  is  $2\omega$ -sectorial (meaning that  $\sigma(T) \subset S_{2\omega}^+ = \{\zeta \in \mathbb{C} : |\arg \zeta| \leq 2\omega \text{ or } \zeta = 0\}$  and appropriate resolvent bounds hold) and hence  $T^2$  has a unique  $\omega$ -sectorial square root  $\sqrt{T^2}$ . It may or may not happen that  $D(\sqrt{T^2}) = D(T)$  with homogeneous ( $\|\sqrt{T^2}u\| \simeq \|Tu\|$ ) or inhomogeneous ( $\|\sqrt{T^2}u\| + \|u\| \simeq \|Tu\| + \|u\|$ ) equivalence of norms. The determination of such equivalences involves studying the holomorphic functional calculus of  $T$  and proving quadratic estimates.

Let  $T$  be an  $\omega$ -bisectorial operator. Then one has the (possibly non-orthogonal) decomposition  $\mathcal{H} = \mathcal{N}(T) \oplus \overline{\mathcal{R}(T)}$  by a variation of the proof of Theorem 3.8 in Cowling–Doust–McIntosh–Yagi [CDMY]. We denote by  $\text{proj}_{\mathcal{N}(T)}$  the projection from  $\mathcal{H}$  onto  $\mathcal{N}(T)$  along this decomposition. Bisectorial operators admit a functional calculus in the following sense [ADM]. For all  $\mu \in (0, \frac{\pi}{2})$  let  $\Psi(S_\mu^o)$  denote the space of all holomorphic functions  $\psi: S_\mu^o \rightarrow \mathbb{C}$  for which there exist  $\alpha, c > 0$  such that

$$|\psi(\zeta)| \leq c \frac{|\zeta|^\alpha}{1 + |\zeta|^{2\alpha}}$$

for all  $\zeta \in \Psi(S_\mu^o)$ . If  $\mu > \omega$  then for all  $\psi \in \Psi(S_\mu^o)$  one can define the bounded operator

$$\psi(T) = \frac{1}{2\pi i} \oint_\gamma \psi(\zeta) (\zeta I - T)^{-1} d\zeta,$$



where  $\gamma$  is a contour in  $S_\mu^o$  enveloping  $S_\omega$  parametrised anti-clockwise. The integral here is simply defined via Riemann sums and this sum converges absolutely as a consequence of the decay of  $\psi$  coupled with the resolvent bounds of  $T$ . If, in addition, there exists a constant  $C > 0$  such that  $\|\psi(T)\| \leq C \|\psi\|_\infty$  for all  $\psi \in \Psi(S_\mu^o)$ , then we say that  $T$  has a **bounded holomorphic  $S_\mu^o$  functional calculus**.

Define  $\text{Hol}^\infty(S_\mu^o)$  to be the space of all bounded functions  $f: S_\mu^o \cup \{0\} \rightarrow \mathbb{C}$  which are holomorphic on  $S_\mu^o$ . For all  $f \in \text{Hol}^\infty(S_\mu^o)$  there exists a uniformly bounded sequence  $(\psi_n)_{n \in \mathbb{N}}$  in  $\Psi(S_\mu^o)$  which converges to  $f$  on  $S_\mu^o$  in the compact-open topology. If in addition  $T$  has a bounded holomorphic  $S_\mu^o$  functional calculus, then  $\lim_{n \rightarrow \infty} \psi_n(T)$  exists in the strong operator topology on  $\mathcal{L}(\mathcal{H})$  by a modification of the proof of the theorem in Section 5 in McIntosh [Mc11], and we define  $f(T) \in \mathcal{L}(\mathcal{H})$  by  $f(T)u = \lim_n \psi_n(T)u + f(0) \text{proj}_{\mathcal{N}(T)} u$  for all  $u \in \mathcal{H}$ . The bounded operator  $f(T)$  is indeed independent of the choice of the sequence  $(\psi_n)_{n \in \mathbb{N}}$ . We then say that  $T$  has a **bounded holomorphic  $H^\infty(S_\mu^o)$ -functional calculus**.

Define  $\chi^\pm: \mathbb{C} \rightarrow \mathbb{C}$  by

$$\chi^\pm(z) = \begin{cases} 1 & \text{if } \pm \text{Re } z > 0, \\ 0 & \text{if } \pm \text{Re } z \leq 0. \end{cases}$$

Moreover, define  $\text{sgn} = \chi^+ - \chi^-$ . Then  $\chi^\pm, \text{sgn} \in \text{Hol}^\infty(S_\mu^o)$  for all  $\mu \in (0, \frac{\pi}{2})$ .

Next we recall some important facts from Axelsson–Keith–McIntosh [AKM1] regarding quadratic estimates. We consider a triple  $(\Gamma, B_1, B_2)$  of operators in  $\mathcal{H}$ . First, we quote the following hypotheses from this reference.

- (H1) The operator  $\Gamma: D(\Gamma) \rightarrow \mathcal{H}$  is a closed, densely defined operator from  $D(\Gamma) \subset \mathcal{H}$  into  $\mathcal{H}$  such that  $\mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$ . So  $\Gamma^2 = 0$ .
- (H2) The operators  $B_1$  and  $B_2$  are bounded on  $\mathcal{H}$ . Moreover, there are  $\kappa_1, \kappa_2 > 0$  such that

$$\begin{aligned} \text{Re}(B_1 u, u) &\geq \kappa_1 \|u\|^2 \quad \text{for all } u \in \mathcal{R}(\Gamma^*), \text{ and} \\ \text{Re}(B_2 u, u) &\geq \kappa_2 \|u\|^2 \quad \text{for all } u \in \mathcal{R}(\Gamma). \end{aligned}$$

- (H3) The operators  $B_1$  and  $B_2$  satisfy  $B_1 B_2(\mathcal{R}(\Gamma)) \subset \mathcal{N}(\Gamma)$  and  $B_2 B_1(\mathcal{R}(\Gamma^*)) \subset \mathcal{N}(\Gamma^*)$ .

Define  $\Pi = \Gamma + \Gamma^*$  and  $\Pi_B = \Gamma + B_1 \Gamma^* B_2$ . Then  $\Pi$  is self-adjoint. If

$$\omega = \frac{1}{2} \left( \arccos \frac{\kappa_1}{\|B_1\|} + \arccos \frac{\kappa_2}{\|B_2\|} \right) \quad (6)$$

then it follows from Proposition 2.5 in [AKM1] that  $\Pi_B$  is  $\omega$ -bisectorial. Hence for all  $t > 0$  one can define the operator  $Q_t^B = i\Pi_B(I + t^2\Pi_B^2)^{-1} \in \mathcal{L}(\mathcal{H})$ .

The following proposition highlights the connection between the harmonic analysis and bounded holomorphic functional calculus.

**Theorem 2.7 (Kato square root type estimate)** *Suppose the triple  $(\Gamma, B_1, B_2)$  satisfies (H1)–(H3) and the operator  $\Pi_B$  satisfies the quadratic estimate*

$$\int_0^\infty \|Q_t^B u\|^2 \frac{dt}{t} \simeq \|u\|^2$$

*for all  $u \in \overline{\mathcal{R}(\Pi_B)}$ . Then the following hold:*

- I. For all  $\mu \in (\omega, \frac{\pi}{2})$  the operator  $\Pi_B$  has a bounded holomorphic  $S_\mu^o$  functional calculus, where  $\omega$  is as in (6).
- II. The Hilbert space admits the spectral decomposition

$$\mathcal{H} = \mathcal{N}(\Pi_B) \oplus \overline{\mathcal{R}(\Pi_B)} = \mathcal{N}(\Pi_B) \oplus \mathcal{R}(\chi^+(\Pi_B)) \oplus \mathcal{R}(\chi^-(\Pi_B)),$$

where the sums are in general not orthogonal.

- III. One has  $D(\Pi_B) = D(\sqrt{\Pi_B^2})$  and

$$\|\Gamma u\| + \|B_1 \Gamma^* B_2 u\| \simeq \|\Pi_B u\| \simeq \|\sqrt{\Pi_B^2} u\|$$

for all  $u \in D(\Pi_B)$ .

**Proof** ‘I’. This is contained in the proof of Proposition 4.8 in [AKM1].

‘II’. The first equality follows from [AKM1] Proposition 2.2. The second equality follows from the bounded  $H^\infty$  functional calculus of  $\Pi_B$ .

‘III’. This follows from Lemma 4.2 in [AKM1] and again functional calculus. Essentially it uses the fact that  $\text{sgn}(\Pi_B)$  is bounded (by I), along with the identities  $\sqrt{\Pi_B^2} u = \text{sgn}(\Pi_B) \Pi_B u$  and  $\Pi_B u = \text{sgn}(\Pi_B) \sqrt{\Pi_B^2} u$ .  $\square$

### 3 The homogeneous problem

In this section we prove the homogeneous subelliptic Kato problem by an application of the results in Axelsson–Keith–McIntosh [AKM1] and Morris [Mor] (see also Bandara [Ban]). Let  $(\mathcal{X}, d, \mu)$  be the metric measure space and let  $(\Gamma, B_1, B_2)$  be the triple of operators in the Hilbert space  $\mathcal{H}$  as in Subsection 2.2. We recall that for a scalar valued Lipschitz function  $\eta$  we denote by  $\text{Lip } \eta$  the pointwise Lipschitz constant. If no confusion is possible, then we identify a measurable function with the associated multiplication operator.

The hypothesis that are required are as follows.

- (H4) The metric space  $(\mathcal{X}, d)$  is complete and connected. The measure  $\mu$  is Borel-regular and doubling. Moreover, there exists a number  $N \in \mathbb{N}$  such that  $\mathcal{H} = L_2(\mathcal{X}, \mathbb{C}^N; \mu)$ .
- (H5) The operators  $B_1$  and  $B_2$  are multiplication operators by bounded matrix valued functions, denoted again by  $B_1, B_2 \in L_\infty(\mathcal{X}, \mathcal{L}(\mathbb{C}^N))$ .
- (H6) There exists a constant  $C > 0$  such that for every bounded Lipschitz function  $\eta: \mathcal{X} \rightarrow \mathbb{R}$  one has
  - (a) the multiplication operator  $\eta I$  leaves  $D(\Gamma)$  invariant, and,
  - (b) the commutator  $[\Gamma, \eta I]$  is again a multiplication operator satisfying the bounds

$$|([\Gamma, \eta I]u)(x)| \leq C |(\text{Lip } \eta)(x)| |u(x)|$$

for a.e.  $x \in \mathcal{X}$  and all  $u \in D(\Gamma)$ .

(H7) For every open ball  $B$  in  $\mathcal{X}$  one has

$$\int_B \Gamma u \, d\mu = 0 \quad \text{and} \quad \int_B \Gamma^* v \, d\mu = 0$$

for all  $u \in D(\Gamma)$  and  $v \in D(\Gamma^*)$  with  $\text{supp } u \subset B$  and  $\text{supp } v \subset B$ .

(H8) There exist  $C_1, C_2 > 0$ ,  $M \in \mathbb{N}$  and an operator  $Z: D(Z) \subset L_2(\mathcal{X}, \mathbb{C}^N) \rightarrow L_2(\mathcal{X}, \mathbb{C}^M)$ , such that

(a)  $D(\Pi) \cap \mathcal{R}(\Pi) \subset D(Z)$ ,

(b) (coercivity)  $\|Zu\| \leq C_2 \|\Pi u\|$

for all  $u \in D(\Pi) \cap \mathcal{R}(\Pi)$ , and,

(c) (Poincaré estimate)  $\int_B |u - \langle u \rangle_B|^2 \, d\mu \leq C_1 r^2 \int_B |Zu|^2 \, d\mu$

for all  $x \in \mathcal{X}$ ,  $r > 0$  and  $u \in D(\Pi) \cap \mathcal{R}(\Pi)$ , where  $B = B(x, r)$  and  $\langle u \rangle_B := \frac{1}{\mu(B)} \int_B u \, d\mu$ .

Hypothesis (H6) implies that  $\Gamma$  behaves like a first order differential operator.

The required quadratic estimates for the operator  $\Pi_B$  now follow almost from Theorem 2.4 in Bandara [Ban].

**Theorem 3.1** *Suppose  $(\mathcal{X}, d, \mu)$ ,  $\mathcal{H}$  and the triple  $(\Gamma, B_1, B_2)$  satisfy Hypotheses (H1)–(H8). Then the operator  $\Pi_B$  satisfies the quadratic estimate*

$$\int_0^\infty \|Q_t^B u\|^2 \frac{dt}{t} \simeq \|u\|^2$$

for all  $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathcal{H}$ . Hence for all  $\mu \in (\omega, \frac{\pi}{2})$ , the operator  $\Pi_B$  has a bounded  $H^\infty(S_\mu^o)$ -functional calculus, where  $\omega$  is as in (6).

**Proof** The change to (H8) only affects Proposition 5.9 in Bandara [Ban]. This change forces the weighted Poincaré inequality Proposition 5.8 in [Ban] to become

$$\int_{\mathcal{X}} |u(x) - u_Q|^2 \left\langle \frac{d(x, Q)}{t} \right\rangle^{-M} d\mu(x) \lesssim \int_{\mathcal{X}} |t(Zu)(x)|^2 \left\langle \frac{d(x, Q)}{t} \right\rangle^{p-M} d\mu(x)$$

for all  $u \in \mathcal{R}(\Pi) \cap D(\Pi)$ . Consequently, in the proof of Proposition 5.9 in [Ban] we can invoke our coercivity Hypothesis (H8) similar to the proof of Proposition 5.5 in [AKM1] to obtain the same conclusion. The rest of the proof of Theorem 2.4 in [Ban] remains unchanged.  $\square$

For the proof of Theorem 1.2 we apply Theorem 3.1. Recall that in this case,  $G$  is the local direct product of a connected compact Lie group and a connected nilpotent Lie group.

**Proof of Theorem 1.2** Choose  $\mathcal{X} = G$ , with Haar measure and distance being the subelliptic distance. Let  $\mathcal{H} = L_2(G) \oplus L_2(G, \mathbb{C}^m) = L_2(G, \mathbb{C}^{1+m})$ . Define  $D(\Gamma) = W'_{1,2}(G) \oplus L_2(G, \mathbb{C}^m)$  and define  $\Gamma: D(\Gamma) \rightarrow \mathcal{H}$  by

$$\Gamma = \begin{pmatrix} 0 & 0 \\ \nabla & 0 \end{pmatrix}.$$

Next let  $B$  be the multiplication operator on  $L_2(G, \mathbb{C}^m)$  by bounded matrix valued functions  $(b_{kl})$ . Define  $B_1, B_2: \mathcal{H} \rightarrow \mathcal{H}$  by

$$B_1 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

Then

$$\Gamma^* = \begin{pmatrix} 0 & -\operatorname{div} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Pi = \begin{pmatrix} 0 & -\operatorname{div} \\ \nabla & 0 \end{pmatrix}.$$

Note that  $\Pi$  is self-adjoint. We next verify that Hypotheses (H1)–(H8) are valid.

(H1) Since  $C_c^\infty(G)$  and therefore also  $W'_{1,2}(G)$  is dense in  $L_2(G)$ , and all the  $A_k$  are closed operators, this hypothesis is obvious.

(H2) Clearly  $B_1$  and  $B_2$  are bounded. First note that

$$\operatorname{Re}(B_1(f, 0), (f, 0)) = \int_G b |f|^2 \geq \kappa_1 \|f\|_{L_2(G)}^2 = \kappa_1 \|(f, 0)\|_{\mathcal{H}}^2$$

for all  $f \in L^2(G)$ , and hence when  $f = -\operatorname{div} w$  for all  $w \in D(\operatorname{div})$ .

Second, if  $w \in W'_{1,2}(G)$ , then

$$\operatorname{Re}(B_2(0, \nabla w), (0, \nabla w)) = \operatorname{Re} \int_G \langle B \nabla w, \nabla w \rangle \geq \kappa \int_G |\nabla w|^2 = \kappa \|(0, \nabla w)\|_{\mathcal{H}}^2,$$

where we used the Gårding inequality (2).

(H3) This trivially holds, since  $B_1 B_2 = B_2 B_1 = 0$ .

(H4) The metric space  $(G, d)$  is complete by Proposition 2.3.II. By assumption,  $G$  is connected. The Haar measure is Borel-regular by Sections 11 and 15 in [HR]. Since  $G$  is the local direct product of a connected compact Lie group and a connected nilpotent Lie group it has polynomial growth. Therefore the metric measure space  $(G, d, \mu)$  has the doubling property by Proposition 2.4.II.

(H5) This is obvious.

(H6) Let  $\eta$  be a bounded real valued Lipschitz function on  $G$ . Then  $\eta \in \bigcap_{k=1}^m D(A_k^{(\infty)})$  by Proposition 2.5.I. Let  $u = (u_1, u_2) \in D(\Gamma)$ . Then  $u_1 \in W'_{1,2}(G)$  and therefore  $\eta u_1 \in W'_{1,2}(G)$ . So  $\eta u \in D(\Gamma)$ . Moreover, for a.e.  $x \in G$  one has

$$\begin{aligned} ([\Gamma, \eta I]u)(x) &= (0, (\nabla(\eta u_1))(x)) - (0, (\eta \nabla u_1)(x)) \\ &= (0, (A_1^{(\infty)}\eta)(x) u_1(x), \dots, (A_m^{(\infty)}\eta)(x) u_m(x)). \end{aligned}$$

Hence  $|([\Gamma, \eta I]u)(x)| \leq (\operatorname{Lip} \eta)(x) |u(x)|$  by Proposition 2.5.II.

(H7) Let  $B$  be an open ball in  $G$  and let  $u = (u_1, u_2) \in D(\Gamma)$  with  $\text{supp } u \subset B$ . There exists a function  $\chi \in C_c^\infty(B)$  such that  $\chi(x) = 1$  for all  $x \in \text{supp } u$ . Then

$$\begin{aligned} \int_B \Gamma u &= (0, \int_B \nabla u_1) = (0, \int_B (\nabla u_1) \bar{\chi}) = (0, (A_1 u_1, \chi), \dots, (A_m u_1, \chi)) \\ &= -(0, (u_1, A_1 \chi), \dots, (u_1, A_m \chi)) = (0, 0). \end{aligned}$$

Similarly, let  $v = (v_1, v_2) \in D(\Gamma^*)$  with  $\text{supp } v \subset B$ . There exists a function  $\chi \in C_c^\infty(B)$  such that  $\chi(x) = 1$  for all  $x \in \text{supp } v$ . Then

$$\begin{aligned} \int_B \Gamma^* v &= (- \int_B \text{div } v_2, 0) = (- \int_B \text{div } v_2 \bar{\chi}, 0) = (-(\text{div } v_2, \chi)_{L_2(G)}, 0) \\ &= ((v_2, \nabla \chi)_{L_2(G, \mathbb{C}^m)}, 0) = (0, 0). \end{aligned}$$

(H8) Define the operator  $Z: D(Z) \rightarrow L_2(G, \mathbb{C}^m) \oplus L_2(G, \mathbb{C}^{m^2}) \simeq L_2(G, \mathbb{C}^{m+m^2})$  by  $D(Z) = W'_{1,2}(G) \oplus W'_{1,2}(G, \mathbb{C}^m) \subset \mathcal{H}$  and

$$Z(u_1, u_2) = (\nabla u_1, \tilde{\nabla} u_2)$$

where  $\tilde{\nabla}$  is defined in (4). Let  $(u_1, u_2) \in D(\Pi) \cap \mathcal{R}(\Pi)$ . Then there exists an element  $(v_1, v_2) \in D(\Pi)$  such that  $(u_1, u_2) = \Pi(v_1, v_2)$ . This implies that  $u_1, v_1 \in W'_{1,2}(G)$ ,  $u_2, v_2 \in D(\text{div})$  and, moreover,  $(u_1, u_2) = (-\text{div } v_2, \nabla v_1)$ . Therefore  $\nabla v_1 = u_2 \in D(\text{div})$  and hence

$$v_1 \in D(\text{div } \nabla) = D(\Delta) = \bigcap_{k,l=1}^m D(A_k A_l)$$

by Proposition 2.1. So  $u_2 = \nabla v_1 \in W'_{1,2}(G, \mathbb{C}^m)$ . This implies that  $(u_1, u_2) \in D(Z)$ . We proved that  $D(\Pi) \cap \mathcal{R}(\Pi) \subset D(Z)$ . Moreover, if  $C_2 \geq 1$  is as in Proposition 2.1 then

$$\begin{aligned} \|Z(u_1, u_2)\|^2 &= \|\nabla u_1\|^2 + \|\tilde{\nabla} u_2\|^2 = \|\nabla u_1\|^2 + \sum_{k,l=1}^m \|A_k A_l v_1\|^2 \\ &\leq C_2 (\|\nabla u_1\|^2 + \|\Delta v_1\|^2) \\ &= C_2 (\|\nabla u_1\|^2 + \|\text{div } u_2\|^2) \\ &= C_2 \|\Pi(u_1, u_2)\|^2. \end{aligned}$$

Finally, the Poincaré estimate follows from (P.1) in Saloff-Coste–Stroock [SS] (page 118).

Now it follows from Theorem 3.1 that the operator  $\Pi_B$  has a bounded  $H^\infty$  functional calculus. Since  $\text{sgn} \in \text{Hol}^\infty(S_\mu^o)$  for all  $\mu \in (0, \frac{\pi}{2})$ , one has  $D(\Pi_B) = D(\sqrt{\Pi_B^2})$  and  $\|\Pi_B u\| \simeq \|\sqrt{\Pi_B^2} u\|$  for all  $u \in D(\Pi_B)$  by Theorem 2.7.III. Note that

$$\Pi_B = \begin{pmatrix} 0 & -b \text{div } B \\ \nabla & 0 \end{pmatrix} \quad \text{and} \quad \Pi_B^2 = \begin{pmatrix} bH & 0 \\ 0 & \tilde{H} \end{pmatrix},$$

where  $\tilde{H}u = -\nabla(b \operatorname{div}(Bu))$  for all  $u \in L_2(G, \mathbb{C}^m)$  with  $(0, u) \in D(\Pi_B^2)$ . Then

$$\sqrt{\Pi_B^2} = \begin{pmatrix} \sqrt{bH} & 0 \\ 0 & \sqrt{\tilde{H}} \end{pmatrix}$$

and  $\|\Pi_B(u_1, u_2)\|^2 = \|\nabla u_1\|^2 + \|b \operatorname{div}(Bu_2)\|^2$  for all  $(u_1, u_2) \in D(\Pi_B) = W'_{1,2}(G) \oplus D(\operatorname{div} \circ B)$ . Restricting to the scalar valued functions gives  $D(\sqrt{bH}) = W'_{1,2}(G)$  and  $\|\sqrt{bH}u\| \simeq \|\nabla u\|$ . This completes the proof of Theorem 1.2.  $\square$

We conclude this section with a proof of the homogeneous estimates of Theorem 1.3 for homogeneous strongly elliptic operators on connected Lie groups with polynomial growth.

**Proof of Theorem 1.3** We use the structure theory for Lie groups with polynomial growth as developed in Dungey–ter Elst–Robinson [DER] and summarised on pages 125–126. There exists another group multiplication  $*$  on  $G$  such that the manifold  $G$  with multiplication  $*$  is a Lie group, denoted by  $G_N$ , which is the local direct product of a connected compact Lie group and a connected nilpotent Lie group. Let  $\mathfrak{g}_N$  be the Lie algebra of  $G_N$ . Then  $\mathfrak{g}_N = \mathfrak{g}$  as vector spaces. The Haar measure  $\mu$  on  $G$  is again a Haar measure on  $G_N$ . Moreover,  $W_{1,2}(G, a) = W_{1,2}(G_N, a)$ . There exist a Lie group homomorphism  $\bar{\mathcal{S}}: G_N \rightarrow \operatorname{Aut}(\mathfrak{g}_N)$ , an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}_N$  and an orthonormal basis  $b_1, \dots, b_m$  of  $\mathfrak{g}_N$  such that  $\bar{\mathcal{S}}(x)$  is orthogonal on  $\mathfrak{g}_N$  for all  $x \in G$  and

$$(A_k u)(x) = \sum_{l=1}^m \langle \bar{\mathcal{S}}(x) b_l, a_k \rangle (B_l^{(N)} u)(x)$$

for a.e.  $x \in G$ ,  $k \in \{1, \dots, m\}$  and  $u \in W_{1,2}(G, a)$ , where  $B_l^{(N)}$  is the infinitesimal generator of  $G_N$  in the direction  $b_l$ . Note that  $W_{1,2}(G, a) = W_{1,2}(G_N, a) = W_{1,2}(G_N, b)$  since both  $a_1, \dots, a_m$  and  $b_1, \dots, b_m$  are vector space bases in  $\mathfrak{g}$ . Moreover, there exists a constant  $C \geq 1$  such that

$$\frac{1}{C} \sum_{k=1}^m \|A_k u\|^2 \leq \sum_{k=1}^m \|B_k^{(N)} u\|^2 \leq C \sum_{k=1}^m \|A_k u\|^2 \quad (7)$$

for all  $u \in W_{1,2}(G, a)$ .

Now

$$H = - \sum_{k', l'=1}^m B_{k'}^{(N)} \tilde{c}_{k' l'} B_{l'}^{(N)}$$

as a divergence form operator, where

$$\tilde{c}_{k' l'}(x) = \sum_{k, l=1}^m \langle \bar{\mathcal{S}}(x) b_{k'}, a_k \rangle c_{kl}(x) \langle \bar{\mathcal{S}}(x) b_{l'}, a_l \rangle.$$

Using (7) it follows that  $H$  satisfies the assumptions of Theorem 1.2 with respect to the coefficients  $\tilde{c}_{kl}$ , the group  $G_N$  and the basis  $b_1, \dots, b_m$ . Therefore  $\|\sqrt{bH}u\| \simeq \sum_{k=1}^m \|B_k^{(N)} u\|$  by Theorem 1.2. A further use of the equivalence (7) of the seminorms, completes the proof of the theorem.  $\square$

## 4 The inhomogeneous problem

To solve the inhomogeneous problem, we apply the results on quadratic estimates in the framework of Morris [Mor], with appropriate modifications, in particular to Hypotheses (H6) and (H8) of [Mor].

For us, this means we continue to use Hypotheses (H1)–(H3) from Section 2, and Hypotheses (H5), (H6) from Section 3, together with the following ones.

(H4i) The metric measure space  $(\mathcal{X}, d, \mu)$  is a complete, connected, locally exponentially doubling metric measure space with a Borel-regular measure  $\mu$ . Moreover, there exists a number  $N \in \mathbb{N}$  such that  $\mathcal{H} = L_2(\mathcal{X}, \mathbb{C}^N; \mu)$ .

(H7i) There exists a constant  $c > 0$  such that for every open ball  $B$  in  $\mathcal{X}$  with radius at most 1 one has

$$\left| \int_B \Gamma u \, d\mu \right| \leq c \mu(B)^{\frac{1}{2}} \|u\| \quad \text{and} \quad \left| \int_B \Gamma^* v \, d\mu \right| \leq c \mu(B)^{\frac{1}{2}} \|v\|$$

for all  $u \in D(\Gamma)$  and  $v \in D(\Gamma^*)$  with  $\text{supp } u \subset B$  and  $\text{supp } v \subset B$ .

(H8i) There exist  $C_1, C_2 > 0$ ,  $M \in \mathbb{N}$  and an operator  $Z: D(Z) \rightarrow L_2(\mathcal{X}, \mathbb{C}^M)$  with  $D(Z) \subset L_2(\mathcal{X}, \mathbb{C}^N) = \mathcal{H}$ , such that

(a)  $D(\Pi) \cap \mathcal{R}(\Pi) \subset D(Z)$ ,

(b) (coercivity)  $\|Zu\| + \|u\| \leq C_2 \|\Pi u\|$

for all  $u \in D(\Pi) \cap \mathcal{R}(\Pi)$ , and,

(c) (Poincaré estimate)  $\int_B |u - \langle u \rangle_B|^2 \, d\mu \leq C_1 r^2 \int_B (|Zu|^2 + |u|^2) \, d\mu$

for all  $x \in \mathcal{X}$ ,  $r \in (0, \infty)$  and  $u \in D(\Pi) \cap \mathcal{R}(\Pi)$ , where  $B = B(x, r)$ .

**Remark 4.1** For any  $r_0 > 0$ , the Poincaré estimate in (H8i) is valid uniformly for all  $r \geq r_0$ .

We are now able to formulate the theorem on quadratic estimates for the inhomogeneous operators.

**Theorem 4.2** *Suppose  $(\mathcal{X}, d, \mu)$ ,  $\mathcal{H}$  and the triple  $(\Gamma, B_1, B_2)$  satisfy Hypotheses (H1), (H2), (H3), (H4i), (H5), (H6), (H7i) and (H8i). Then the operator  $\Pi_B$  satisfies the quadratic estimate*

$$\int_0^\infty \|Q_t^B u\|^2 \frac{dt}{t} \simeq \|u\|^2$$

for all  $u \in \overline{\mathcal{R}(\Pi_B)} \subset L_2(\mathcal{X}, \mathbb{C}^N)$ . Hence  $\Pi_B$  has a bounded  $H^\infty$ -functional calculus.

**Proof** First, we note that for all non-empty subsets  $E, F$  on any metric space  $X$  satisfying  $d(E, F) > 0$  one can find a Lipschitz function  $\eta: X \rightarrow [0, 1]$  such that  $\eta = 1$  on  $E$ ,  $\eta = 0$  on  $F$ , and **Lip**  $\eta \leq 1/d(E, F)$ , where **Lip**  $\eta$  is the Lipschitz constant of  $\eta$ . Also observe that all the smooth cutoff functions used in Morris [Mor] can be replaced by Lipschitz equivalents, in particular allowing us to obtain off-diagonal estimates in the present case.

Next, our alteration to (H6) and (H8) allows us to dispense with the use of the Sobolev spaces with respect to the Levi–Civita connection in [Mor] and simply consider  $\mathcal{R}(\Pi) \cap D(\Pi)$ . Explicitly, the weighted Poincaré inequality in Lemma 5.7 of [Mor] is altered to read  $\|Zu\|$  instead of  $\|\nabla u\|$ , and by coercivity, Proposition 5.8 in [Mor] holds. Then Proposition 5.2 in [Mor] holds, since we still have  $\|u\| \lesssim \|\Pi u\|$ . Finally, we observe that the measure merely needs to be locally exponentially doubling and Borel-regular.  $\square$

Now we are able to prove Theorem 1.1, using ideas from [Mor], which have their roots in [AKM2].

**Proof of Theorem 1.1** We apply Theorem 4.2 with  $\mathcal{X} = G$  and  $\mathcal{H} = L_2(G) \oplus (L_2(G) \oplus L_2(G, \mathbb{C}^m)) = L_2(G, \mathbb{C}^{2+m})$ . Define  $D(\Gamma) = W'_{1,2}(G) \oplus L_2(G) \oplus L_2(G, \mathbb{C}^m)$  and define  $\Gamma: D(\Gamma) \rightarrow \mathcal{H}$  by

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ \nabla & 0 & 0 \end{pmatrix}.$$

Let  $B$  be the multiplication operator on  $L_2(G) \oplus L_2(G, \mathbb{C}^m) = L_2(G, \mathbb{C}^{1+m})$  by bounded matrix valued functions

$$\begin{pmatrix} b_0 & b'_1 & \cdots & b'_m \\ b_1 & & & \\ \vdots & (b_{kl}) & & \\ b_m & & & \end{pmatrix}.$$

Next define  $B_1, B_2: \mathcal{H} \rightarrow \mathcal{H}$  by

$$B_1 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

Then

$$\Gamma^* = \begin{pmatrix} 0 & I & -\text{div} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Pi = \begin{pmatrix} 0 & I & -\text{div} \\ I & 0 & 0 \\ \nabla & 0 & 0 \end{pmatrix}.$$

Note again that  $\Pi$  is self-adjoint.

The proof that Hypotheses (H1), (H3), (H5) and (H6) are valid is similar to the proof of Theorem 1.2 in Section 3. Also (H2) follows similarly from the Gårding inequality (1). It remains to verify Hypotheses (H4i), (H7i) and (H8i).

(H4i) The only difference with (H4) is the locally exponentially doubling property, which follows from Proposition 2.4.I.

(H7i) Let  $B$  be an open ball in  $G$  and let  $u = (u_1, u_2, u_3) \in D(\Gamma)$  with  $\text{supp } u \subset B$ . Then  $\int_B \nabla u_1 = 0$  by the argument in the proof of (H7) in Section 3. Therefore

$$\left| \int_B \Gamma u \right|^2 = \left| \int_B u_1 \right|^2 + \left| \int_B \nabla u_1 \right|^2 = \left| \int_B u_1 \right|^2 \leq \mu(B) \|u_1\|^2 \leq \mu(B) \|u\|^2.$$



Similarly, if  $v = (v_1, v_2, v_3) \in D(\Gamma^*)$  with  $\text{supp } v \subset B$  then  $\int_B \text{div } v_3 = 0$  and

$$\left| \int_B \Gamma^* v \right| = \left| \int_B v_2 - \text{div } v_3 \right| = \left| \int_B v_2 \right| \leq \mu(B)^{1/2} \|v\|$$

as required.

(H8i) Define the operator  $Z: D(Z) \rightarrow L_2(G, \mathbb{C}^m) \oplus L_2(G, \mathbb{C}^m) \oplus L_2(G, \mathbb{C}^{m^2}) = L_2(G, \mathbb{C}^{2m+m^2})$  by  $D(Z) = W'_{1,2}(G) \oplus W'_{1,2}(G) \oplus W'_{1,2}(G, \mathbb{C}^m) \subset \mathcal{H}$  and

$$Z(u_1, u_2, u_3) = (\nabla u_1, \nabla u_2, \tilde{\nabla} u_3).$$

Let  $u = (u_1, u_2, u_3) \in D(\Pi) \cap \mathcal{R}(\Pi)$ . Then there exists an element  $(v_1, v_2, v_3) \in D(\Pi)$  such that  $(u_1, u_2, u_3) = \Pi(v_1, v_2, v_3) = (v_2 - \text{div } v_3, v_1, \nabla v_1)$  with in particular,  $u_1, u_2 = v_1 \in W'_{1,2}(G)$  and  $u_3, v_3 \in D(\text{div})$ . Therefore  $\nabla v_1 = u_3 \in D(\text{div})$  and hence

$$v_1 \in D(\text{div } \nabla) = D(\Delta) = \bigcap_{k,l=1}^m D(A_k A_l)$$

by Proposition 2.1. So  $u_3 = \nabla v_1 \in W'_{1,2}(G, \mathbb{C}^m)$ . This implies that  $u = (u_1, u_2, u_3) \in D(Z)$ . We proved that  $D(\Pi) \cap \mathcal{R}(\Pi) \subset D(Z)$ , and it remains for us to obtain the bound  $\|Zu\| + \|u\| \leq C_2 \|\Pi u\|$ .

Note that

$$\|Zu\|^2 = \|\nabla u_1\|^2 + \|\nabla u_2\|^2 + \|\tilde{\nabla} u_3\|^2$$

and

$$\|\Pi u\|^2 = \|u_2 - \text{div } u_3\|^2 + \|u_1\|^2 + \|\nabla u_1\|^2 = \|(I - \Delta)v_1\|^2 + \|u_1\|^2 + \|\nabla u_1\|^2.$$

Clearly

$$\|\nabla u_2\|^2 = \|\nabla v_1\|^2 = (\Delta v_1, v_1) \leq \|\Delta v_1\| \|v_1\| \leq \|(I - \Delta)v_1\|^2$$

and if  $C_1 \geq 1$  as in Proposition 2.1, then

$$\|\tilde{\nabla} u_3\|^2 = \sum_{k,l=1}^m \|A_k A_l v_1\|^2 \leq C_1 (\|\Delta v_1\|^2 + \|v_1\|^2) \leq 2C_1 \|(I - \Delta)v_1\|^2.$$

So

$$\|Z(u_1, u_2, u_3)\|^2 \leq 3C_1 \|\Pi(u_1, u_2, u_3)\|^2.$$

Next we show that  $\|(u_1, u_2, u_3)\|^2 \leq (4C_1 m^2 + 4) \|\Pi(u_1, u_2, u_3)\|^2$ . Trivially,  $\|u_1\|^2 \leq \|\Pi(u_1, u_2, u_3)\|^2$ . Moreover,

$$\|\text{div } u_3\|^2 \leq m^2 \|\tilde{\nabla} u_3\|^2 \leq 2C_1 m^2 \|(I - \Delta)v_1\|^2$$

and

$$\|u_2\|^2 \leq 2\|u_2 - \text{div } u_3\|^2 + 2\|\text{div } u_3\|^2 \leq (4C_1 m^2 + 2) \|(I - \Delta)v_1\|^2.$$

Also  $\|u_3\|^2 = \|\nabla v_1\|^2 \leq \|(I - \Delta)v_1\|^2$ .

We conclude that  $\|Zu\| + \|u\| \leq C_2 \|\Pi u\|$  (for a suitable constant  $C_2$ ) as required.

Finally we prove the Poincaré inequality. Jerison [Jer] proved that there exist  $c, r_0 > 0$  such that

$$\int_{B(r)} |f - \langle f \rangle_{B(r)}|^2 \leq c r^2 \int_{B(r)} |\nabla f|^2 \quad (8)$$

for all  $r \in (0, r_0]$  and  $f \in C^\infty(\overline{B(r)})$ . Since  $C^\infty(G) \cap W'_{1,2}(G)$  is dense in  $W'_{1,2}(G)$  by Lemma 2.4 in ter Elst–Robinson [ER1], it follows that (8) is valid for all  $r \in (0, r_0]$  and  $f \in W'_{1,2}(G)$ . Let  $R$  denote the right regular representation in  $G$  and let  $\delta$  be again the modular function on  $G$ . Then

$$\begin{aligned} \int_{B(x,r)} |f - \langle f \rangle_{B(x,r)}|^2 &= \delta(x) \int_{B(r)} |R(x)f - \langle R(x)f \rangle_{B(r)}|^2 \\ &\leq c \delta(x) r^2 \int_{B(r)} |\nabla R(x)f|^2 = c r^2 \int_{B(x,r)} |\nabla f|^2 \end{aligned}$$

for all  $x \in G$ ,  $r \in (0, r_0]$  and  $f \in W'_{1,2}(G)$ . The Poincaré estimate follows by Remark 4.1.

Now one can complete the proof of Theorem 1.1 similarly as in Section 3 by an application of Theorem 4.2.  $\square$

## 5 Further results

We have chosen to take the (left) Haar measure  $\mu$  on  $L_2(G; \mu)$ , and the infinitesimal generators are with respect to the left regular representation in  $L_2(G; \mu)$ . Another option would be to choose the right Haar measure  $\nu$  on  $G$  and consider the left regular representation in  $L_2(G; \nu)$ . Then the solution to the Kato problem has the following formulation.

**Theorem 5.1** *Let  $a_1, \dots, a_m$  be an algebraic basis for the Lie algebra  $\mathfrak{g}$  of a connected Lie group  $G$ . For all  $k \in \{1, \dots, m\}$  let  $A_k^{(R)}$  be the infinitesimal generator of the one-parameter group  $t \mapsto L^{(R)}(\exp ta_k)$ , where  $L^{(R)}$  denotes the left regular representation in  $L_2(G; \nu)$ . For all  $k, l \in \{1, \dots, m\}$  let  $b_{kl}, b_k, b'_k, b_0 \in L_\infty(G)$ . Assume there exist  $\kappa, c_1 > 0$  such that*

$$\operatorname{Re} \sum_{k,l=1}^m (b_{kl} A_k^{(R)} u, A_l^{(R)} u) \geq \kappa \sum_{k=1}^m \|A_k^{(R)} u\|^2 - c_1 \|u\|^2$$

for all  $u \in \bigcap_{k=1}^m D(A_k^{(R)})$ . Consider the divergence form operator

$$H = \sum_{k,l=1}^m (A_k^{(R)})^* b_{kl} A_l^{(R)} + \sum_{k=1}^m b_k A_k^{(R)} + \sum_{k=1}^m (A_k^{(R)})^* b'_k + b_0 I$$

in  $L_2(G; \nu)$ , where the norm and inner product are in  $L_2(G; \nu)$ . Suppose  $\operatorname{Re} b_0$  is large enough such that  $-H$  generates a bounded semigroup on  $L_2(G; \nu)$ . Let  $b \in L_\infty(G)$  and suppose there exists a constant  $\kappa_1 > 0$  such that  $\operatorname{Re} b \geq \kappa_1$  a.e. Then

$$D(\sqrt{bH}) = \bigcap_{k=1}^m D(A_k^{(R)})$$

with equivalent norms.

**Proof** The proof is almost the same as the proof of Theorem 1.1, so we indicate the differences. We replace all  $A_k$  by  $A_k^{(R)}$ . Note that for all  $k \in \{1, \dots, m\}$  there exists a constant  $\beta_k \in \mathbb{R}$  such that  $(A_k^{(R)})^* = -A_k^{(R)} + \beta_k I$ , where the adjoint is in  $L_2(G; \nu)$ . We take the same subelliptic distance on  $G$ . There exists a constant  $c > 0$  such that  $\nu = c\delta^{-1}\mu$ , where  $\delta$  is the modular function. (See [HR] Theorem 15.15.) Moreover, since  $\delta$  is a continuous homomorphism, there exist  $M, \omega > 0$  such that  $\delta(x) \leq M e^{\omega d(x,e)}$  for all  $x \in G$ . Hence by Proposition 2.3 there are  $c, C, \lambda > 0$  and  $D' \in \mathbb{N}$  such that  $c r^{D'} \leq \nu(B(r)) \leq C r^{D'}$  for all  $r \in (0, 1]$  and  $\nu(B(r)) \leq C e^{\lambda r}$  for all  $r \geq 1$ . (Actually, the natural number  $D'$  is the same as in Proposition 2.3.III.) Then Hypothesis (H4i) follows. Next consider Hypothesis (H7i). Let  $B$  be an open ball in  $G$ , let  $u = (u_1, u_2, u_3) \in D(\Gamma)$  and suppose that  $\text{supp } u \subset B$ . There exists a function  $\chi \in C_c^\infty(B)$  such that  $\chi(x) = 1$  for all  $x \in \text{supp } u$ . Let  $k \in \{1, \dots, m\}$ . Then

$$\int_B A_k^{(R)} u_1 d\nu = (u_1, (A_k^{(R)})^* \bar{\chi})_{L_2(G; \nu)} = (u_1, (-A_k^{(R)} + \beta_k I) \bar{\chi})_{L_2(G; \nu)} = \beta_k \int_B u_1 d\nu.$$

So  $|\int_B \nabla u_1 d\nu| \leq \sqrt{\beta_1^2 + \dots + \beta_m^2} \nu(B)^{1/2} \|u\|$ . The rest of the proof of Hypothesis (H7i) is similar.

All other hypothesis have the same proof as before.  $\square$

One can also consider the infinitesimal generators with respect to the right regular representation on  $L_2(G; \mu)$  or  $L_2(G; \nu)$ . Then the inhomogeneous Kato problem has again a solution. This follows from Theorems 1.1 and 5.1 by using the inversion  $x \mapsto x^{-1}$  on  $G$ . We leave the formulation of the two theorems to the reader.

## 6 Stability

Finally we consider stability under holomorphic perturbation.

Let  $U \subset \mathbb{C}$  be an open set,  $\omega \in [0, \frac{\pi}{2})$  and for all  $\zeta \in U$  let  $T(\zeta)$  be an  $\omega$ -bisectorial operator in  $\mathcal{H}$  with domain  $D(T(\zeta))$ . Let  $\mu \in (\omega, \frac{\pi}{2})$ . We say that  $T$  has a **uniformly bounded holomorphic  $H^\infty(S_\mu^o)$ -functional calculus** if there exists a constant  $C > 0$  such that  $\|\psi(T(\zeta))\| \leq C \|\psi\|_\infty$  uniformly for all  $\psi \in \Psi(S_\mu^o)$  and  $\zeta \in U$ .

**Theorem 6.1** *Let  $U \subset \mathbb{C}$  be an open set,  $\mathcal{H}$  a Hilbert space and  $(X, d, \mu)$  a metric measure space. Let  $B_1, B_2: U \rightarrow \mathcal{L}(\mathcal{H})$  be bounded holomorphic functions. Suppose that the triple  $(\Gamma, B_1(\zeta), B_2(\zeta))$  satisfies (H1)–(H8) uniformly for all  $\zeta \in U$ , with constants  $\kappa_1$  and  $\kappa_2$ . Let*

$$\omega = \sup_{\zeta \in U} \frac{1}{2} \left( \arccos \frac{\kappa_1}{\|B_1(\zeta)\|} + \arccos \frac{\kappa_2}{\|B_2(\zeta)\|} \right) < \frac{\pi}{2}.$$

*Let  $\mu \in (\omega, \frac{\pi}{2})$ . Then one has the following.*

- I.** *The operator  $\Pi_{B(\zeta)}$  is  $\omega$ -bisectorial operator in  $\mathcal{H}$  uniformly for all  $\zeta \in U$ .*
- II.** *The family  $\zeta \mapsto \Pi_{B(\zeta)}$  has a uniformly bounded holomorphic  $H^\infty(S_\mu^o)$ -functional calculus.*
- III.** *For all  $f \in \text{Hol}^\infty(S_\mu^o)$  the map  $\zeta \mapsto f(\Pi_{B(\zeta)})$  is holomorphic.*

**Proof** Statement I follows from Proposition 2.5 in Axelsson–Keith–McIntosh [AKM1] and Statement II from Theorem 3.1. Statement III follows as in the proof of Theorem 6.4 in [AKM1].  $\square$

We conclude the paper by noting the following stability result.

**Theorem 6.2** *Let  $G$  be the local direct product of a connected compact Lie group and a connected nilpotent Lie group. Let*

$$H = - \sum_{k,l=1}^m A_k b_{kl} A_l$$

*be a homogeneous divergence form operator with bounded measurable coefficients satisfying the subellipticity condition (2) with constant  $\kappa_1$ . Let  $b \in L_\infty(G)$  and suppose there exists a constant  $\kappa_2 > 0$  such that  $\operatorname{Re} b \geq \kappa_2$  a.e. Let  $\eta_1 \in (0, \kappa_1)$  and  $\eta_2 \in (0, \kappa_2)$ . Then there exists a constant  $C > 0$  such that the following is valid. For all  $k, l \in \{1, \dots, m\}$  let  $\tilde{b}_{kl} \in L_\infty(G)$  and suppose that*

$$\widetilde{M} = \sup_{x \in G} \|(\tilde{b}_{kl}(x))\|_{\mathbb{C}^{m \times m}} \leq \eta_1.$$

*Further, let  $\tilde{b} \in L_\infty(G)$  with  $\|\tilde{b}\|_\infty \leq \eta_2$ . Let*

$$\tilde{H} = - \sum_{k,l=1}^m A_k (b_{kl} + \tilde{b}_{kl}) A_l.$$

*Then*

$$\|\sqrt{(b + \tilde{b})\tilde{H}}u - \sqrt{bH}u\| \leq C(\widetilde{M} + \|\tilde{b}\|_\infty)\|\nabla u\|$$

*for all  $u \in W'_{1,2}(G)$ .*

**Proof** This follows as in the proof of Theorem 6.5 in [AKM1], using Theorems 3.1 and Theorem 6.1 with  $B(\zeta) = B + \zeta\tilde{B}$  for all  $\zeta \in U$ , where  $U$  is an appropriate open set with  $[0, 1] \subset U \subset \mathbb{C}$ . See also the proof of Theorem 7.2 in [BMc].  $\square$

There are similar stability results for the inhomogeneous problems as in Theorem 1.1, or with the right Haar measure or right translations. We leave the formulation to the reader.

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## References

- [ADM] ALBRECHT, D., DUONG, X. and MCINTOSH, A., Operator theory and harmonic analysis. In *Instructional Workshop on Analysis and Geometry, Part III*, vol. 34 of Proceedings of the Centre for Mathematics and its Applications. Australian National University, Canberra, 1996, 77–136.
- [AHLMT] AUSCHER, P., HOFMANN, S., LACEY, M., MCINTOSH, A. and TCHAMITCHIAN, P., The solution of the Kato square root problem for second order elliptic operators on  $\mathbb{R}^n$ . *Ann. of Math. (2)* **156** (2002), 633–654.
- [AT] AUSCHER, P. and TCHAMITCHIAN, P., Square root problem for divergence operators and related topics. *Astérisque* **249** (1998), viii+172.
- [AKM1] AXELSSON, A., KEITH, S. and MCINTOSH, A., Quadratic estimates and functional calculi of perturbed Dirac operators. *Invent. Math.* **163** (2006), 455–497.
- [AKM2] ———, The Kato square root problem for mixed boundary value problems. *J. London Math. Soc. (2)* **74** (2006), 113–130.
- [Ban] BANDARA, L., Quadratic estimates for perturbed Dirac type operators on doubling measure metric spaces, 2011 (Submitted). arXiv:1107.3905.
- [BMc] BANDARA, L., MCINTOSH, A., The Kato square root problem on vector bundles with generalised bounded geometry, 2012 (Submitted). arXiv:1203.0373.
- [Car] CARATHÉODORY, C., Untersuchungen über die Grundlagen der Thermodynamik. *Math. Anal.* **67** (1909), 355–386.
- [CDMY] COWLING, M., DOUST, I., MCINTOSH, A. and YAGI, A., Banach space operators with a bounded  $H^\infty$  functional calculus. *J. Austr. Math. Soc. (Series A)* **60** (1996), 51–89.
- [DER] DUNGEY, N., ELST, A. F. M. TER and ROBINSON, D. W., *Analysis on Lie groups with polynomial growth*, vol. 214 of Progress in Mathematics. Birkhäuser Boston Inc., Boston, 2003.
- [ER1] ELST, A. F. M. TER and ROBINSON, D. W., Subcoercive and subelliptic operators on Lie groups: variable coefficients. *Publ. RIMS. Kyoto Univ.* **29** (1993), 745–801.
- [ER2] ———, Subelliptic operators on Lie groups: regularity. *J. Austr. Math. Soc. (Series A)* **57** (1994), 179–229.
- [ER3] ———, Subcoercivity and subelliptic operators on Lie groups II: The general case. *Potential Anal.* **4** (1995), 205–243.
- [ERS] ELST, A. F. M. TER, ROBINSON, D. W. and SIKORA, A., Riesz transforms and Lie groups of polynomial growth. *J. Funct. Anal.* **162** (1999), 14–51.

- [Gui] GUIVARC'H, Y., Croissance polynomiale et périodes des fonctions harmoniques. *Bull. Soc. Math. France* **101** (1973), 333–379.
- [Hei] HEINONEN, J., *Lectures on analysis and metric spaces*. Universitext. Springer, New York, 2001.
- [HR] HEWITT, E. and ROSS, K. A., *Abstract harmonic analysis I*. Second edition, Grundlehren der mathematischen Wissenschaften 115. Springer-Verlag, Berlin etc., 1979.
- [Hof] HOFMANN, S., A short course on the Kato problem. In *Second Summer school in analysis and mathematical physics (Cuernavaca, 2000)*, vol. 289 of Contemp. Math., 61–77. Amer. Math. Soc., Providence, RI, 2001.
- [Jer] JERISON, D., The Poincaré inequality for vector fields satisfying Hörmander's condition. *Duke Math. J.* **53** (1986), 503–523.
- [McI1] MCINTOSH, A., Operators which have an  $H_\infty$  functional calculus. In JEFFERIES, B., MCINTOSH, A. and RICKER, W. J., eds., *Miniconference on operator theory and partial differential equations*, vol. 14 of Proceedings of the Centre for Mathematical Analysis. CMA, ANU, Canberra, Australia, 1986, 210–231.
- [McI2] MCINTOSH, A., The square root problem for elliptic operators. In *Functional analytic methods for partial differential equations (Tokyo, 1989)*, Lecture Notes in Mathematics 1450. Springer-Verlag, Berlin etc., 1990, 122–140.
- [Mor] MORRIS, A. J., The Kato square root problem on submanifolds *J. London Math. Soc. (2)* (2012). To appear, arXiv:1103.5089.
- [NSW] NAGEL, A., STEIN, E. M. and WAINGER, S., Balls and metrics defined by vector fields I: basic properties. *Acta Math.* **155** (1985), 103–147.
- [SS] SALOFF-COSTE, L. and STROOCK, D. W., Opérateurs uniformément sous-elliptiques sur les groupes de Lie. *J. Funct. Anal.* **98** (1991), 97–121.
- [VSC] VAROPOULOS, N. T., SALOFF-COSTE, L. and COULHON, T., *Analysis and geometry on groups*. Cambridge Tracts in Mathematics 100. Cambridge University Press, Cambridge, 1992.

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